# Homotopy groups of spheres

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**Homotopy groups of spheres** is a branch of mathematics, specifically algebraic topology, that attempts to understand the different ways spheres of various dimensions can be wrapped around each other. The topic can be hard to understand because the most interesting and surprising results involve spheres in higher dimensions. An $n$-dimensional sphere, ***n*-sphere** or hypersphere, is a generalization of the familiar circle (1-sphere) and sphere (2-sphere) and is defined as all the points in a space of $n+1$ dimensions that are a fixed distance from a center point.

Mappings are continuous functions from one sphere into another. Two mappings are considered equivalent if one can be continuously deformed into the other. One sphere can always be mapped into another by assigning all the points on the first sphere to a single point on the second. This mapping, and all those equivalent to it, are called the trivial mapping and are represented by 0. Mappings of a $i$-sphere ($i$ here is another integer; we'll need $n$ in a moment) into any topological space, $X$, can be combined by splicing them together, roughly speaking, and the equivalence classes of these mappings ("homotopy classes") form a group called the $i^{\text{th}}$ homotopy group of $X$, which is written $\pi_i\ (X)$.

Homotopy groups of spheres studies the cases where $X$ is an $n$-sphere, and the groups are written $\pi_i\ (S^n)$.

The goal of algebraic topology is to categorize or classify topological spaces. Homotopy groups were invented in the late 19th century[1] (http://www-history.mcs.st-and.ac.uk/~history/Biographies/Jordan.html) as a tool for such classification, in effect using the set of mappings from an $n$-sphere in to a space as a way to probe the structure of that space. An obvious question was how this new tool would work on $n$-spheres themselves. No general solution to this question has been found to date, but many homotopy groups of spheres have been computed and the results are surprisingly rich and complicated.

## Contents

## Cases that are easy to visualize

The few cases that can be constructed in ordinary 3-dimensional space allow some sense of the subject to be gained. These visualizations are not, of course, mathematical proofs.

### $\pi_1(S^1)$

The simplest case considers the ways a circle (1-sphere) can be wrapped around another circle. This can be visualized by wrapping a rubber band around one's finger. One can wrap it once, twice, three times and so on. One can do the wrapping in either of two directions and if one wraps a certain number of times in one direction and then wraps the same number of time in the other direction, then wraps in opposite directions will cancel out. The group that is formed is therefore the group of integers, known as the infinite cyclic group, and denoted **Z**.

Elements of π1(S1)

Elements of $\pi_1(S^1)$

### $\pi_2(S^2)$

The case of 2-spheres can be visualized as wrapping a plastic bag around an ordinary toy ball and then sealing it. The sealed bag is topologically equivalent to a 2-sphere, as is the surface of the ball. The bag can be wrapped more than once by twisting it and wrapping it back over the ball. (The bag is allowed to pass through itself). The twist can be in one of two directions and opposite twists can cancel out. Again the group of integers, **Z**, is formed. These two results generalize: for all $n > 0$, $\pi_n(S^n) = \mathbf{Z}$.

### $\pi_1(S^2)$

Any continuous mapping from a circle to an ordinary sphere can be continuously deformed down to a point, i.e. the trivial mapping. This can be visualized as a rubber-band wrapped around a frictionless ball - it can always be slid off. Further, this result generalises for higher dimensions. All mappings from a hypersphere into a sphere of higher dimension are similarly trivial: if $i < n$, then $\pi_i(S^n) = 0$ (i.e., the trivial group).

### $\pi_2(S^1)$

All the interesting cases of homotopy groups of spheres involve mappings from an *n*-sphere dimensional onto spheres of lower dimensions. Unfortunately, the one such case we might easily visualize is uninteresting, as there are no non-trivial mappings from the ordinary sphere to the circle. Hence, $\pi_2(S^1) = 0$.

# Higher dimensions

The rest of this article provides a summary of the homotopy groups for spheres for higher dimensions. Computing the homotopy group can be complicated, and this article is restricted to summarizing many of the homotopy groups of spheres that have been computed to date. The results are surprisingly complex, with no pattern discerned to date.

From a geometric point of view homotopy groups are fundamental invariants. From the algebraic aspect, there is ample evidence that they involve substantial complexity of structure, and intense study from around 1950 has not completely elucidated that.

### $\pi_3(S^2)$

As mentioned above, when $i$ is less than $n$, the homotopy group is the trivial group: $\pi_i\ (S^n) = 0$, and the case $i = n$ is always the infinite cyclic group, $\pi_n\ (S^n) = \mathbf{Z}$. It is the case $i > n$ that is of real importance, and historically it came as a great surprise that the corresponding homotopy groups were nontrivial. This is in contrast to the behavior for corresponding results in homology theory, where $H_i\ (S^n) = 0$ when $i > n$. The first nontrivial example concerned mappings from the three-dimensional sphere to the ordinary 2-sphere, and was discovered by Heinz Hopf in 1931. The existence of the Hopf fibration implies that $\pi_3\ (S^2) = \mathbf{Z}$.

### Stable and unstable groups

As the homotopy groups of spheres turn out to be very difficult to compute, algebraic topologists searched for ways to simplify the problem. A key insight was that the suspension theorem of Hans Freudenthal implies that the groups $\pi_{n+k}\ (S^n)$ depend only on $k$ for $n \geq k + 2$. These groups are called the **stable homotopy groups of spheres**, and are denoted $\pi_k{}^S$. They are finite and abelian for $k \geq 1$. They have been computed in numerous cases, but the general pattern is still elusive.

For $n < k + 2$, the groups are called the **unstable homotopy groups of spheres**. There are ad-hoc methods of calculating these for the cases of $n$ small. A systematic tool in this context is the J-homomorphism. These groups are abelian, and are all finite except for those of the form $\pi_{4r-1}\ (S^{2r})$ (for integer values of $r$). In this case, the group is the product of the infinite cyclic group with a finite abelian group.

# Table of homotopy groups

Tables of homotopy groups of spheres are most conveniently organized by only showing $\pi_{n+k}(S^n)$ for $n > 1$ and $k > 0$. For other cases, the homotopy group $\pi_{n+k}(S^n)$, with $n > 0$, is as follows:

- For $k < 0$, the group is the trivial group (generally written 0)
- For $k = 0$, the group is the infinite cyclic group (generally written **Z**)
- For $k > 0$ and $n = 1$, the group is trivial.

The following table shows many of the groups $\pi_{n+k}(S^n)$ that have been computed to date, with the following conventions:

- Where the entry is an integer, $m$, the homotopy group is the cyclic group of that order (generally written $\mathbf{Z}_m$).
- Where the entry is "Z", the homotopy group is the infinite cyclic group, (**Z**).
- Where entry is a sum, the homotopy group is the direct sum (equivalently, cartesian product) of the cyclic groups of those orders. Powers indicate repeated summation.

Example: $\pi_{19}(S^{10}) = \pi_{10+9}(S^{10}) = Z + 2^3$, the table entry at $k = 9$, $n = 10$. The expression $Z + 2^3$ denotes the group $\mathbf{Z} \times \mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2$.

| | ***n* = 2** | ***n* = 3** | ***n* = 4** | ***n* = 5** | ***n* = 6** | ***n* = 7** | ***n* = 8** | ***n* = 9** | ***n* = 10** | ***n* = 11** | ***n* = 12** | ***n* > *k* + 1** |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| ***k* = 1** | Z | | | | | | | | | | | 2 |
| ***k* = 2** | 2 | 2 | | | | | | | | | | 2 |
| ***k* = 3** | 2 | 12 | Z + 12 | | | | | | | | | 24 |
| ***k* = 4** | 12 | 2 | $2^2$ | 2 | | | | | | | | 0 |
| ***k* = 5** | 2 | 2 | $2^2$ | 2 | Z | | | | | | | 0 |
| ***k* = 6** | 2 | 3 | 24 + 3 | 2 | 2 | 2 | | | | | | 2 |
| ***k* =** | 3 | 15 | 15 | 30 | 60 | 120 | Z + 120 | | | | | 240 |

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **7** | 3 | 15 | 15 | 30 | 60 | 120 | Z + 120 | | | | | 240 |
| ***k* = 8** | 15 | 2 | 2 | 2 | 8 + 6 | $2^3$ | $2^4$ | $2^3$ | | | | $2^2$ |
| ***k* = 9** | 2 | $2^2$ | $2^3$ | $2^3$ | $2^3$ | $2^4$ | $2^5$ | $2^4$ | *Z* + $2^3$ | | | 23 |
| ***k* = 10** | $2^2$ | 12 + 2 | $40 + 4 + 2 + 3^2$ | 18 + 8 | 18 + 8 | 24 + 2 | $8^2 + 2 + 3^2$ | 24 + 2 | 12 + 2 | $2^2 + 3$ | | 2 + 3 |
| ***k* = 11** | 12 + 2 | $84 + 2^2$ | $84 + 2^5$ | $504 + 2^2$ | 504 + 4 | 504 + 2 | 504 + 2 | 504 + 2 | 504 | 504 | *Z* + 504 | 504 |
| ***k* = 12** | $84 + 2^2$ | $2^2$ | $2^6$ | $2^3$ | 240 | 0 | 0 | 0 | 4 + 3 | 2 | $2^2$ | *See below* |
| ***k* = 13** | $2^2$ | 6 | $8 + 2^2 + 3^2$ | $2^2 + 3$ | 6 | 6 | $2^2 + 3$ | 6 | 6 | $2^2 + 3$ | $2^2 + 3$ | |
| ***k* = 14** | 6 | 30 | $840 + 9 + 2^2$ | $2^2 + 3$ | 12 + 2 | 24 + 4 | 60 + 48 + 8 | 16 + 4 | 16 + 2 | 16 + 2 | 24 + 16 | |
| ***k* = 15** | 30 | 30 | 30 | $15 + 2^2$ | $10 + 4 + 3^2$ | $120 + 2^3$ | $120 + 2^5$ | $240 + 2^3$ | $240 + 2^2$ | 30 + 16 | 30 + 16 | |
| ***k* = 16** | 30 | $2^2 + 3$ | $2^3 + 3^2$ | $2^2$ | $504 + 2^2$ | $2^4$ | $2^7$ | $2^4$ | 30 + 16 | 2 | 2 | |
| ***k* = 17** | $2^2 + 3$ | $12 + 2^2$ | $8 + 4^2 + 2^2 + 3^2$ | $4 + 2^2$ | $2^4$ | $2^4$ | $2^5 + 3$ | $2^4$ | $2^3$ | $2^3$ | $2^4$ | |
| ***k* = 18** | $12 + 2^2$ | $12 + 2^2$ | $40 + 4 + 2^5 + 3^2$ | $24 + 2^2$ | $8 + 2^2 + 3^2$ | 24 + 2 | $8^2 + 42 + 9$ | 8 + 2 + 3 | $8 + 2^2 + 3$ | 8 + 4 + 2 | $32 + 30 + 4^2$ | |
| ***k* = 19** | $12 + 2^2$ | 132 + 2 | $132 + 2^5$ | 66 + 8 | 264 + 32 | 264 + 2 | 264 + 2 | 264 + 2 | $22 + 3^2$ | $264 + 2^3$ | $264 + 2^5$ | |

| | ***n* = 13** | ***n* = 14** | ***n* = 15** | ***n* = 16** | ***n* = 17** | ***n* = 18** | ***n* = 19** | ***n* = 20** | ***n* > *k* + 1** |
|---|---|---|---|---|---|---|---|---|---|
| ***k* = 12** | 2 | | | | | | | | 0 |
| ***k* = 13** | 6 | Z + 3 | | | | | | | 3 |
| ***k* = 14** | 16 + 2 | 8 + 2 | 4 + 2 | | | | | | $2^2$ |
| ***k* = 15** | 32 + 30 | 32 + 30 | 32 + 30 | Z + 32 + 30 | | | | | $3^2 + 30$ |

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| ***k* = 16** | 2 | 8 + 6 | $2^3$ | $2^4$ | $2^3$ | | | | $2^2$ |
| ***k* = 17** | $2^4$ | $2^4$ | $2^5$ | $2^6$ | $2^5$ | Z + $2^4$ | | | $2^4$ |
| ***k* = 18** | $8^2$ + 2 | $8^2$ + 2 | $8^3$ + 2 + 3 | $8^3$ + 2 + 3 | $8^2$ + 2 | 8 + 6 | 8 + 22 | | 8 + 2 |
| ***k* = 19** | 264 + $2^3$ | 66 + 8 + 4 | 264 + $2^2$ | 8 + $2^2$ + 3 + 11 | 264 + $2^2$ | 66 + 8 | 66 + 8 | Z + 66 + 8 | 132 + 8 |

Note that when *a* and *b* have no common factor, $\mathbf{Z}_a$ x $\mathbf{Z}_b$ is isomorphic to $\mathbf{Z}_{ab}$. Using this, entries in the above have been written using fewer coefficients than in Toda's table.

## See also

- 3-sphere#Topological properties for a table of the homotopy groups $\pi_k$ ($S^3$)
- Winding number

## References


- Allen Hatcher: *Stable homotopy groups of spheres (http://www.math.cornell.edu/%7Ehatcher/AT/ATpage.html)* page 339. (2002, Cambridge University Press). ISBN 0-521-79540-0.
- Table of homotopy groups of spheres (http://www.math.nus.edu.sg/%7Ematwujie/homotopy_groups_sphere.html) . From Hirosi Toda: *Composition Methods in Homotopy Groups of Spheres* (Princeton University. Press, 1962). ISBN 0-691-09586-8.
- Douglas C. Ravenel: *Complex cobordism and stable homotopy groups of spheres* (2nd edition), Appendix 3 (http://www.math.rochester.edu/people/faculty/doug/mybooks/ravenelA3.pdf) . (From online edition (http://www.math.rochester.edu/people/faculty/doug/mu.html) ). (Ams Chelsea Pub., 2003) ISBN 0-8218-2967-X


## Further reading

- relation with string theory (http://www.math.niu.edu/~rusin/known-math/97/homotopy.spheres)

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